

There will be associated with the eigenvalues of matrix  $A$  a number of extraneous eigenvalues which are not roots of Equation (1a); these are easily recognized, however, as they are of order  $\sqrt[n-m]{\Gamma}$  greater than the other eigenvalues of  $\bar{A}$  where  $m$  is the number of roots of Equation (1a). This may be established by comparing coefficients of the characteristic equation of (2a) with the corresponding expansion of  $\det(\bar{A} - \lambda I) = 0$ . A typical value of  $\Gamma$  used to obtain zeros of the system was  $\Gamma = 10^{15}$ .

The gain  $K$  occurring in Equation (5) may be calculated as follows:

Consider the system of Equation (1)

$$\dot{x} = Ax + Bu(t) \quad (4a)$$

when  $u(t)$  is a unit step function and  $t \rightarrow \infty$ . Then

$$\lim_{t \rightarrow \infty} x(t) = -A^{-1}B \quad (5a)$$

provided Equation (1) represents a stable system (that is, eigenvalues of  $A$  are all in the left-hand part of the complex

plane, this is normally the case for the types of equations considered in this paper).

From (5)

$$\lim_{t \rightarrow \infty} x_r(t) = \lim_{s \rightarrow 0} s \chi_r(s)$$

$$= \lim_{s \rightarrow 0} s \frac{K(s - \alpha_1)(s - \alpha_2) \dots (s - \alpha_m)}{(s - \beta_1)(s - \beta_2) \dots (s - \beta_n)} \frac{1}{s} \quad (6a)$$

$$= K(-)^{m-n} \frac{\alpha_1 \alpha_2 \dots \alpha_m}{\beta_1 \beta_2 \dots \beta_n} \quad (7a)$$

Therefore the equating of (7a) and (5a) gives

$$K = (-)^{m-n-1} \frac{\beta_1 \beta_2 \dots \beta_n}{\alpha_1 \alpha_2 \dots \alpha_m} (A^{-1}B)_r \quad (8a)$$

where  $(A^{-1}B)_r$  is the  $r$ th element of the vector  $(A^{-1}B)$ .

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# Multiphase Viscoelastic Flow Through Porous Media

JOHN C. SLATTERY

Mobil Oil Corporation, Dallas, Texas

Local volume averaging of the equations of continuity and of motion over each phase in a porous medium is discussed. For a Noll simple material, inertial effects may be neglected with respect to viscous effects as the product of a Reynolds number with a Weissenberg number goes to zero (in the limit of a Newtonian fluid, inertial effects may be neglected as the Reynolds number goes to zero). When inertial effects can be neglected, a resistance transformation for each phase is introduced which in part maps the local volume-averaged velocity vector into the local force per unit volume which that phase exerts on the pore walls and the moving phase interfaces. Capillary pressure is defined through a local surface average of the jump balance for momentum. The shape of the curve denoting capillary pressure as a function of saturation in a two-phase flow may vary with the contact angle. For an isotropic multiphase flow of incompressible materials, the functional dependence of the resistance coefficient for a Noll simple material and for a Newtonian fluid is discussed by means of the Buckingham-Pi theorem.

In an attempt to treat two-phase flows of Newtonian fluids in a porous medium, Darcy's law is written for each phase  $i$  (1, p. 653),

$$\nabla P^i + \frac{\mu_i}{\alpha_i k} \nabla V^i = 0 \quad (1)$$

Here  $P^i$  and  $V^i$  are understood to represent local averages of pressure and velocity, respectively, over phase  $i$ ;  $\alpha_i$  is termed the relative permeability, which is assumed to be a function of the local saturation of phase  $i$  alone (by saturation of phase  $i$  we mean the volume of phase  $i$  per unit volume of pores);  $k$  is the permeability of the rock to a single phase Newtonian fluid;  $\mu_i$  is the viscosity of phase  $i$ . The local pressures  $P^1$  and  $P^2$  of the two phases

John C. Slattery is also at Northwestern University, Evanston, Ill.

are further assumed to be related by the capillary pressure  $P_c$  defined by

$$P^1 - P^2 = P_c \quad (2)$$

The capillary pressure  $P_c$  is taken to be a function only of the local saturation of phase  $i$ .

An obvious disadvantage to Equations (1) and (2) is that they are proposed by analogy rather than derived. This means that  $P^i$  and  $V^i$  have no unambiguous physical significance and that the assumed functional dependence of the relative permeability  $\alpha_i$  and the capillary pressure  $P_c$  is open to question. This is not a firm basis upon which to study multiphase flow of viscoelastic fluids.

In what follows we first extend our previous treatment (2) of local averaging to multiphase flows; the only assumption made is that the various phases are in equilibrium with one another in the sense that there is no mass transfer across the various phase boundaries. We discuss the conditions under which inertial effects can be neglected with respect to viscous effects in the equation of motion for a Noll simple material. (It is believed that the Noll simple material represents the behavior of a wide range of real viscoelastic materials.) After defining the resistance transformation for phase  $i$  to extend Darcy's law to multiphase flow, we discover that a further relation is necessary. The concept of capillary pressure is motivated by an interfacial average of the jump balance for momentum at a phase interface. We conclude with an examination of the functional dependence of the resistance coefficient in the case of two incompressible Noll simple materials and in the case of two incompressible Newtonian fluids.

## LOCAL AVERAGING

The discussion in this section applies to any number of phases moving relative to one another through some arbitrary geometry. We might be talking about the displacement of oil by water in a porous medium or we might be considering the flow of an emulsion or slurry through a tube. We might have  $L$  phases present or there may be only two, the one dispersed in the other. Our intention is to form a local average of the equations of motion and continuity over one of these phases, phase  $i$ . In this section we shall make only one restriction on the physical situation: the various phases are assumed to be in equilibrium with one another in the sense that there is no mass transfer across the phase interfaces.

Let us associate with a fixed point in space a closed surface  $S$ , the volume of which is  $V$ . This point may lie in the interior of  $S$ , but this is not essential. We may identify the surface  $S$  with every point in space by a translation (without rotation) of  $S$ . For example, if  $S$  is a unit sphere, the center of which coincides with the point considered, we center upon each point in space a unit sphere.

Let us integrate the stress equation of motion over the volume or region  $V_i$  occupied by phase  $i$  in the interior of  $S$

$$\int_{V_i} \left\{ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \rho \mathbf{v} \mathbf{v} - \nabla \cdot \mathbf{t} - \rho \mathbf{f} \right\} dV = 0 \quad (3)$$

Applying the divergence theorem to the second and third terms, we have

$$\begin{aligned} \int_{V_i} \frac{\partial \rho \mathbf{v}}{\partial t} dV + \int_{S_i} \rho \mathbf{v} \mathbf{v} \cdot \mathbf{n} dS \\ - \int_{S_i} \mathbf{t} \cdot \mathbf{n} dS - \int_{V_i} \rho \mathbf{f} dV = 0 \end{aligned} \quad (4)$$

Here we denote by  $S_i$  the closed boundary surface of  $V_i$ . Portions of this surface are impermeable and fixed in space  $S_{fi}$ ; portions denoted by  $S_{mi}$  form boundaries with other moving and deforming phases present; the remainder  $S_{ei}$  are the intersections of phase  $i$  with  $S$ . We may consider  $S_{ei}$  as the entrance and exit surfaces of the system.

The first term on the left of Equation (4) may be rewritten using a generalized form of the transport theorem (3, p. 347) applied to our system:

$$\int_{V_i} \frac{\partial \rho \mathbf{v}}{\partial t} dV = \frac{\partial}{\partial t} \int_{V_i} \rho \mathbf{v} dV - \int_{S_{mi}} \rho \mathbf{v} V_{(n)} dS \quad (5)$$

Here  $V_{(n)}$  is the speed of displacement of the surface  $S_{mi}$  (3, p. 499). By a theorem developed previously [2, Equation (8)] we have for any tensor quantity  $\mathbf{B}$

$$\nabla \cdot \int_{V_i} \mathbf{B} dV = \int_{S_{ei}} \mathbf{B} \cdot \mathbf{n} dS \quad (6)$$

This allows us to write the second and third terms on the left-hand side of Equation (4) as

$$\int_{S_i} \rho \mathbf{v} \mathbf{v} \cdot \mathbf{n} dS = \nabla \cdot \int_{V_i} \rho \mathbf{v} \mathbf{v} dV + \int_{S_{mi}} \rho \mathbf{v} \mathbf{v} \cdot \mathbf{n} dS \quad (7)$$

and

$$\int_{S_i} \mathbf{t} \cdot \mathbf{n} dS = \nabla \cdot \int_{V_i} \mathbf{t} dV + \int_{S_{mi} + S_{fi}} \mathbf{t} \cdot \mathbf{n} dS \quad (8)$$

If we assume that there is no mass transfer across the phase interface, on  $S_{mi}$

$$V_{(n)} = \mathbf{v} \cdot \mathbf{n} \quad (9)$$

After one divides through by  $V$ , Equation (4) may be rearranged using Equations (5), (7), (8), and (9) to yield

$$\frac{\overline{\partial \rho \mathbf{v}}}{\partial t} + \nabla \cdot \overline{\rho \mathbf{v} \mathbf{v}} = \nabla \cdot \overline{\mathbf{t}} + \overline{\rho \mathbf{f}} + \frac{1}{V} \int_{S_{mi} + S_{fi}} \mathbf{t} \cdot \mathbf{n} dS \quad (10)$$

By the  $\overline{\phantom{x}}$  we mean an average over  $V$  of those quantities associated with phase  $i$ . Equation (10) should be looked upon as a volume average of the equation of motion in terms of volume-averaged variables.

We may volume average the equation of continuity in the same way. If we integrate over  $V_i$ , we have

$$\int_{V_i} \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} \right\} dV = 0 \quad (11)$$

We may rearrange the first term by means of the generalized transport theorem (3, p. 499):

$$\int_{V_i} \frac{\partial \rho}{\partial t} dV = \frac{\partial}{\partial t} \int_{V_i} \rho dV - \int_{S_{mi}} \rho V_{(n)} dS \quad (12)$$

and the second term by means of the divergence theorem and Equation (6):

$$\int_{V_i} \nabla \cdot \rho \mathbf{v} dV = \nabla \cdot \int_{V_i} \rho \mathbf{v} dV + \int_{S_{mi}} \rho \mathbf{v} \cdot \mathbf{n} dS \quad (13)$$

After one divides by  $V$ , Equation (11) yields, by means of Equations (9), (12), and (13)

$$\frac{\overline{\partial \rho}}{\partial t} + \nabla \cdot \overline{\rho \mathbf{v}} = 0 \quad (14)$$

Equation (14) should be interpreted as a volume average

of the equation of continuity in terms of volume-averaged variables.

## CRITERIA FOR NEGLECTING INERTIAL EFFECTS IN A NOLL SIMPLE MATERIAL

Let us focus our attention on a particular material particle in a body which has been undergoing a deformation as a function of time. Let  $\mathbf{X}$  be the position of the material particle in some reference configuration of the body. The motion of the body as a function of time may be described by specifying the positions  $\mathbf{x}$  of all material particles in the body as a function of time:

$$\mathbf{x} = \chi(\mathbf{X}, t) \quad (15)$$

The deformation gradient  $\mathbf{F}$  is a second-order tensor field defined by

$$\mathbf{F} \equiv \nabla_{\mathbf{X}}(\mathbf{X}, t) \quad (16)$$

All constitutive equations for stress which have been successful in describing the behavior of real materials have satisfied what Noll terms the principle of determinism for stress (the stress in the body is determined by the history of the motion of that body) and the principle of local action (in determining the stress at a given material particle, the motion outside an arbitrary neighborhood of the particle may be disregarded) (4, p. 56). A *simple material* (4, p. 61) is defined by Noll to be one for which the stress  $\mathbf{t}$  at  $\mathbf{x}$  and at time  $t$  is determined by the history of the deformation gradient  $\mathbf{F}$  for the material which is in an arbitrarily small neighborhood of  $\mathbf{x}$  at time  $t$ :

$$\mathbf{t} = \frac{\mu_0}{t_0} \overset{\circ}{\mathbf{G}}_{\sigma^*=0}(\mathbf{F}(t - t_0, \sigma^*)) \quad (17)$$

Here we follow Truesdell's discussion of the dimensional invariance of the definition of a simple material (5; 4, p. 65). The quantity  $\overset{\circ}{\mathbf{G}}_{\sigma^*=0}$  is a dimensionally invariant tensor-valued functional, that is, an operator which maps tensor-valued functions into tensors. The constants  $\mu_0$  and  $t_0$  are, respectively, a characteristic viscosity and a characteristic time of the material. Although Truesdell (5) suggests specific definitions for these quantities, for the purposes of our discussion here it is not necessary to make particular choices.

Since the definition of a Noll simple material is so general, it is expected that a wide range in behavior by real viscoelastic materials can be described by this theory. (Viscoelastic is used in the sense that the materials obey neither of the classical linear relations: Newton's law of viscosity and Hooke's law of elasticity.)

Let us put the volume-averaged equation of motion for phase  $i$ , Equation (10), in dimensionless form:

$$\frac{L}{t_p v_i} \frac{\partial \rho^* \mathbf{v}^*}{\partial t} + \nabla \cdot \rho^* \mathbf{v}^* \mathbf{v}^* = \frac{g L}{v_i^2} \rho^* \mathbf{f}^* + \frac{1}{N_{Re}} \frac{1}{N_{Wi}} \left\{ \nabla^* \cdot \mathbf{t}^* + \frac{1}{V^*} \int_{S_{mi}^* + S_{fi}^*} \mathbf{t}^* \cdot \mathbf{n} dS^* \right\} \quad (18)$$

where

$$\begin{aligned} \rho &= \rho_i \rho^*, & \mathbf{v} &= v_i \mathbf{v}^*, & t &= t_p t^* \\ \mathbf{t} &= \frac{\mu_i}{t_i} \mathbf{t}^*, & \mathbf{f} &= g \mathbf{f}^*, & \mathbf{x} &= L \mathbf{x}^* \end{aligned}$$

$$N_{St} = t_p v_i / L, \quad N_{Re} = L v_i \rho_i / \mu_i, \quad N_{Wi} = t_i v_i / L \quad (19)$$

The quantities  $\rho_i$ ,  $\mu_i$ ,  $t_i$ , and  $v_i$  are, respectively, a characteristic density, characteristic viscosity, characteristic

time, and characteristic velocity for phase  $i$ ;  $L$  is a characteristic length of the geometry;  $g$  is the acceleration of gravity; and  $t_p$  is a characteristic time of the process. We have made the stress dimensionless with respect to  $\mu_i/t_i$  as suggested by Equation (17). The quantities  $N_{St}$ ,  $N_{Re}$ , and  $N_{Wi}$  are known respectively as the Strouhal, Reynolds, and Weissenberg numbers. The Weissenberg number has only recently been introduced in discussions of viscoelastic behavior (6, 7).

Equation (18) suggests that, as the product of the Reynolds number with the Weissenberg number goes to zero, we should be able to neglect the convective inertial terms, the second term on the left of Equation (18), with respect to the viscous terms. In a nonperiodic flow it seems reasonable to set  $N_{St} = 1$  by defining  $t_p = L/v_i$ , which suggests that, in the limit as  $N_{Re} N_{Wi} \rightarrow 0$ , we may be able to neglect the local acceleration as well as the convective inertial terms with respect to the viscous terms to obtain

$$\nabla \cdot \frac{\mathbf{i}}{t} + \frac{\mathbf{i}}{\rho \mathbf{f}} + \frac{1}{V} \int_{S_{mi} + S_{fi}} \mathbf{t} \cdot \mathbf{n} dS = 0 \quad (20)$$

One must, of course, use caution in applying these results, since there is no theorem which guarantees that a seemingly arbitrarily small term in a differential equation has a negligible effect on the solution of the differential equation (8, p. 4). Two classic examples of the failure of such an argument are: dropping all viscous terms in the Navier-Stokes equation as the Reynolds number becomes unbounded, and dropping all inertial terms in the Navier-Stokes equation as the Reynolds number approached zero for a two-dimensional flow in orthogonal, Cartesian coordinates.

A special case of the Noll simple material is the incompressible Noll simple fluid with fading memory. In the limit as the Weissenberg number goes to zero for an incompressible Noll simple fluid with fading memory, we have an incompressible Newtonian fluid and the dependence upon the Weissenberg number drops out of the dimensionless equation of motion (7, Equations 9 and 15). This is consistent with the usual argument for Newtonian fluids that, as the Reynolds number goes to zero, inertial effects become negligible with respect to viscous effects (except in the case of two-dimensional flow in orthogonal, Cartesian coordinates).

In multiphase flow through a porous medium, Haines (9) observed that at any given instant a phase interface is undergoing a rapid translation or jump over a relatively short distance in a small fraction of the pores. It seems clear that the inertial terms are not negligibly small with respect to the viscous terms in the equation of motion [the integrand of Equation (3)] for any fluid undergoing a Haines jump. But the argument presented in this section for neglecting inertial terms with respect to viscous terms in the *averaged* equation of motion still appears reasonable, so long as the averaging surface  $S$  includes a sufficiently large number of pores.

## MODIFIED DARCY'S LAW

Our intention in this section is to develop from first principles an equation similar to Equation (1). We restrict ourselves to the consideration of the multiphase flow of incompressible, viscoelastic materials through porous rocks. We assume that the flow is such that the inertial terms can be neglected with respect to the viscous terms in the volume-averaged equation of motion to obtain Equation (20).

Let us assume that our body force vector is represented by a potential

$$\mathbf{f} = -\nabla\phi \quad (21)$$

in order that, assuming incompressibility, we may write Equation (20) as

$$\nabla \bar{\mathcal{P}} - \frac{1}{V} \int_{s_{mi}+s_{fi}} [\mathbf{t} - \rho_i \phi \mathbf{I}] \cdot \mathbf{n} \, dS - \nabla \cdot \frac{\mathbf{i}}{\tau} = 0 \quad (22)$$

Here we define the modified pressure  $\mathcal{P}$  for any particular phase  $i$  as

$$\mathcal{P} = p + \rho_i \phi \quad (23)$$

and the extra stress tensor  $\tau$  as

$$\tau = \mathbf{t} + p \mathbf{I}$$

The pressure  $p$  is defined by requiring

$$\text{tr } \tau = \sum_{i=1}^N \tau_{ii} = 0$$

The  $\tau_{ii}$  are the diagonal components of the extra stress tensor in an arbitrary orthogonal, Cartesian coordinate system. It is easy to show that Equation (22) may also be written as

$$\nabla [\bar{\mathcal{P}} - p_o] - \frac{1}{V} \int_{s_{mi}+s_{fi}} \{\mathbf{t} + [p_o - \rho_i \phi] \mathbf{I}\} \cdot \mathbf{n} \, dS - \nabla \cdot \frac{\mathbf{i}}{\tau} = 0 \quad (24)$$

where  $p_o$  is a constant ambient or reference pressure.

As in our previous discussion of single-phase flow of viscoelastic fluids through porous media (2), we express the last two terms on the left of Equation (22) as the result of a resistance transformation  $\mathbf{K}_i$  acting upon the average velocity vector appropriate to phase  $i$ ,  $\frac{\mathbf{i}}{\mathbf{v}}$ :

$$-\frac{1}{V} \int_{s_{mi}+s_{fi}} \{\mathbf{t} + [p_o - \rho_i \phi] \mathbf{I}\} \cdot \mathbf{n} \, dS - \nabla \cdot \frac{\mathbf{i}}{\tau} = \mathbf{K}_i \cdot \frac{\mathbf{i}}{\mathbf{v}} \quad (25)$$

This allows us to replace Equation (24) by

$$\nabla [\bar{\mathcal{P}} - p_o] + \mathbf{K}_i \cdot \frac{\mathbf{i}}{\mathbf{v}} = 0. \quad (26)$$

By exactly the same arguments as given previously in discussing single-phase flow of viscoelastic fluids through porous media (2), we suggest that  $\mathbf{K}_i$  is invertible and that, if we are dealing with a randomly deposited although perhaps layered porous structure,  $\mathbf{K}_i$  is symmetric and positive-definite as well. A positive-definite transformation is such that its proper values are all real, positive, and nonzero (10, p. 153). Note that we are not suggesting that for an arbitrary anisotropic porous media the resistance transformation should be symmetric.

If there are no preferred directions for the movement of phase  $i$ , the three proper values of  $\mathbf{K}_i$  are equal and denoted by  $K_i$ , a real, positive, nonzero number termed the resistance coefficient for phase  $i$ . Consequently, we may write Equation (26) as

$$\nabla [\bar{\mathcal{P}} - p_o] + K_i \frac{\mathbf{i}}{\mathbf{v}} = 0 \quad (27)$$

We speak of the flow of the phase  $i$  as being isotropic.

Before going further, it is worth noting that  $[\bar{\mathcal{P}} - p_o]$  is not the type of average pressure of phase  $i$  at a point in terms of which experimentalists usually think. They are more likely to consider something close to

$$\langle \bar{\mathcal{P}} - p_o \rangle^i \equiv \frac{1}{V_i} \int_{V_i} [\bar{\mathcal{P}} - p_o] \, dV = \frac{[\bar{\mathcal{P}} - p_o]^i}{s_i \psi} \quad (28)$$

If there are  $N$  fluid phases present in the porous media, we define the porosity  $\psi$  as

$$\psi \equiv \frac{1}{V} \sum_{j=1}^N V_j$$

and the saturation of phase  $i$  as

$$s_i \equiv \frac{V_i}{\sum_{j=1}^N V_j} \quad (29)$$

In a static situation, experimentalists expect

$$\nabla \langle \bar{\mathcal{P}} - p_o \rangle^i = \nabla \langle \bar{\mathcal{P}} \rangle^i = 0$$

since they know that  $\mathcal{P}$  should be a constant  $p_o$  throughout the fluid. But the static limit of Equations (26) and (27)

$$\nabla [\bar{\mathcal{P}} - p_o]^i = 0$$

appears reasonable as well.

Our next step is to discuss the functional dependence of  $K_i$  for an isotropic flow of phase  $i$  [the results for the isotropic case can easily be extended to the anisotropic case as suggested by our previous discussion of single-phase viscoelastic flow through porous media (2)]. In order to prepare ourselves for this discussion, it is necessary to consider the implications of the jump balance for momentum at a phase interface.

## CAPILLARY PRESSURE

In the sections which follow, we limit our considerations to the movement of two incompressible materials, phase  $i$  and phase  $j$ , through a porous rock structure. Under these conditions the volume-averaged equation of continuity for phase  $i$ , Equation (14), reduces to

$$\psi \frac{\partial s_i}{\partial t} + \nabla \cdot \frac{\mathbf{i}}{\mathbf{v}} = 0 \quad (30)$$

Here  $\psi$  is the porosity of the porous medium, assumed to be a constant. The saturation of phase  $i$ ,  $s_i$ , is defined by Equation (29), which implies that

$$s_i + s_j = 1 \quad (31)$$

It is worthwhile at this point to count our unknowns in order to see whether we have a sufficient number of equations with which to determine them. Our unknowns are

ten:  $s_i$ ,  $s_j$ ,  $[\bar{\mathcal{P}} - p_o]^i$ ,  $[\bar{\mathcal{P}} - p_o]^j$ ; the three components of  $\frac{\mathbf{i}}{\mathbf{v}}$  and the three components of  $\frac{\mathbf{j}}{\mathbf{v}}$ . Relating these unknowns, we have Equation (31) as well as for each phase the equation of continuity and the three components of Equation (28). This means that we have nine equations in ten unknowns, which suggests that another equation is required.

As a matter of fact, another equation is not always required. Consider a steady state, parallel flow of two phases through a porous medium such that  $s_i$  and  $s_j$  are independent of position as well as time and specified by a boundary condition. This leaves us with eight equations

and eight unknowns. But since we wish to discuss a more general situation as well, we must look for a further relationship linking these variables.

We take the jump condition for stress at a phase interface to be of the form (11 to 13)

$$\mathbf{t}_{(i)} \cdot \mathbf{n}_{(i)} + \mathbf{t}_{(j)} \cdot \mathbf{n}_{(j)} = -2H\sigma \mathbf{n}_{(i)} \quad (32)$$

By  $\mathbf{t}_{(i)}$  we mean the stress tensor appropriate to phase  $i$  evaluated at the phase boundary;  $\mathbf{n}_{(i)}$  is the unit normal vector to the phase interface pointing into phase  $i$ . The mean curvature of the surface is noted by  $H$  (14, p. 205);  $\sigma$  is the surface tension. In writing Equation (32) we assume that there is no mass transfer at the phase interface. We do not account for any effects due to the deformation of the material in the neighborhood of the phase interface, since these effects are not well understood at the present time (10). With the introduction of the extra stress  $\tau$ , Equation (32) becomes

$$[-p_{(i)} + p_{(j)}] \mathbf{n}_{(i)} + \tau_{(i)} \cdot \mathbf{n}_{(i)} + \tau_{(j)} \cdot \mathbf{n}_{(j)} = -2H\sigma \mathbf{n}_{(i)} \quad (33)$$

In terms of dimensionless variables we have

$$\frac{p_o L}{\sigma} [-p_{(i)}^* + p_{(j)}^*] \mathbf{n}_{(i)} + \frac{\mu_i L}{t_i \sigma} \tau_{(i)}^* \cdot \mathbf{n}_{(i)} + \frac{\mu_j L}{t_j \sigma} \tau_{(j)}^* \cdot \mathbf{n}_{(j)} = -2H^* \mathbf{n}_{(i)}, \quad (34)$$

where

$$p_{(i)} = p_o p_{(i)}^*, \quad \tau_{(i)} = \frac{\mu_i}{t_i} \tau_{(i)}^* \\ H = H^* / L \quad (35)$$

Here  $p_o$  is taken to be a characteristic pressure, and  $\sigma$  is assumed to be a constant.

In order to get a feeling for the magnitudes of the various terms involved in Equation (34), let us consider an example using some typical values for the various characteristic parameters.

#### Example for Viscoelastic Fluids

$$L = 5 \times 10^{-5} \text{ cm.}, \quad t_i = 10^{-2} \text{ sec.}, \\ \mu_i = 10^{-2} \text{ g./ (cm.) (sec.)}, \quad \sigma = 1 \text{ dyne/cm.}, \\ \frac{\mu_i L}{t_i \sigma} = 5 \times 10^{-5} \ll 1$$

We take as our characteristic length the square root of the salt water permeability for a 250 md rock; a characteristic time of  $10^{-2}$  sec. appears to be reasonable for some viscoelastic fluids (15, 16). This suggests that the last two terms on the left-hand side of Equation (34) can often be safely neglected with respect to the term on the right to obtain from Equation (33):

$$p_{(i)} - p_{(j)} = 2H\sigma \quad (36)$$

The argument used to arrive at Equation (36) goes a little differently when one is dealing with Newtonian fluids, since the characteristic time for a Newtonian fluid is zero (5). For a Newtonian fluid the extra stress  $\tau$  is given by

$$\tau = 2\mu \mathbf{d} \quad (37)$$

where the rate-of-deformation tensor  $\mathbf{d}$  is defined as

$$\mathbf{d} = \frac{1}{2} [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \quad (38)$$

Here  $(\nabla \mathbf{v})^T$  is the transpose of  $\nabla \mathbf{v}$ . Equations (37) and (38) suggest that we define a dimensionless extra stress tensor  $\tau^*$  by

$$\tau = \frac{\mu v_i}{L} \tau^* \quad (39)$$

In view of Equation (39), Equation (34) is replaced by

$$\frac{p_o L}{\sigma} [-p_{(i)}^* + p_{(j)}^*] \mathbf{n}_{(i)} + \frac{\mu_i v_i}{\sigma} \tau_{(i)}^* \cdot \mathbf{n}_{(i)} + \frac{\mu_j v_j}{\sigma} \tau_{(j)}^* \cdot \mathbf{n}_{(j)} = -2H^* \mathbf{n}_{(i)} \quad (40)$$

An example shows that we are often justified in neglecting the second and third terms on the left of Equation (40) with respect to the right-hand side.

#### Example for Newtonian Fluids

$$\mu_i = 10^{-2} \text{ g./ (cm.) (sec.)}, \quad \sigma = 1 \text{ dyne/cm.}, \\ v_i = 1 \text{ ft./day or } 3.5 \times 10^{-4} \text{ cm./sec.}$$

$$\frac{\mu_i v_i}{\sigma} = 3.5 \times 10^{-6} \ll 1$$

Under these circumstances we are again left with Equation (36).

Let us average Equation (36) over that portion of the interfacial boundary which is common to the two fluid phases and which is enclosed by  $V$ :

$$\frac{1}{S_{mi}} \int_{S_{mi}} [p_{(i)} - p_{(j)} - 2H\sigma] dS = 0 \quad (41)$$

It seems reasonable to say that the difference between the volume-averaged pressures appropriate to the two phases at a point should be proportional to the interfacial average of the jump in pressure across the phase interface:

$$\langle p - p_o \rangle^i - \langle p - p_o \rangle^j = \langle p \rangle^i - \langle p \rangle^j \\ = \frac{A}{S_{mi}} \int_{S_{mi}} [p_{(i)} - p_{(j)}] dS \quad (42)$$

The volume averaging used here is defined in Equation (28); it appears to be appropriate for comparison with experimental measurements of pressure. Equations (41) and (42) suggest that

$$\langle p - p_o \rangle^i - \langle p - p_o \rangle^j = \frac{A}{S_{mi}} \int_{S_{mi}} 2H\sigma dS \quad (43)$$

The proportionality factor  $A$  may be considered to be a function of the saturation of phase  $i$ ; certainly the area of the interfacial surface common to the fluid phases is a function of the saturation. The mean curvature  $H$  is affected by the contact angle  $\theta$ . In view of these considerations, Equation (43) suggests that we regard the local difference between the volume-averaged pressures appropriate to the two phases as being a function of a characteristic length of the porous media, the surface tension, the contact angle, and the saturation of phase  $i$ ; this function is known as the capillary pressure  $p_c$  (1, p. 632).

$$\langle p - p_o \rangle^i - \langle p - p_o \rangle^j = p_c = p_c(L, \sigma, \theta, s_i) \quad (44)$$

If we apply the Buckingham-Pi theorem (17) to the capillary pressure  $p_c$ , we have for geometrically similar porous media

$$\frac{p_c L}{\sigma} = f(\theta, s_i) \quad (45)$$

Since this functional relationship does not depend upon the local flow conditions (such as velocity or pressure gradient), it may be studied under static conditions by standard methods (18, p. 374).

A typical capillary pressure curve (19, Figure 1) for imbibition (reentry of water into a water-wet porous rock containing oil) shows that a certain amount of oil is left in the rock. One purpose of adding surfactants to the water might be to modify the shape of the capillary pressure curve as a function of saturation in order to leave less oil behind. Equation (45) shows that the shape of the dimensionless capillary pressure curve as a function of saturation is controlled by the contact angle  $\theta$ , which implies that the modification of the contact angle is of primary concern here. In the case of water flooding a porous structure containing crude oil and naturally occurring surfactants, this suggests that the shape of the capillary pressure curve might best be altered by control of the pH and salinity of the water (20).

Equations (28), (31), and (44), Equation (29) for each phase, and the three components of Equation (28) for each phase, furnish us with twelve equations in twelve

unknowns:  $s_i, s_j, \overline{[\mathcal{P} - p_o]^i}, \overline{[\mathcal{P} - p_o]^j}, <\mathcal{P} - p_o>^i, <\mathcal{P} - p_o>^j$ ; the three components of  $\overline{\mathbf{v}}^i$ ; and the three components of  $\overline{\mathbf{v}}^j$ .

### THE FUNCTIONAL DEPENDENCE OF THE RESISTANCE COEFFICIENT

Let us discuss the functional dependence of the resistance coefficient for phase  $i$ ,  $K_i$  in Equation (28). We continue to restrict ourselves to two incompressible fluid phases passing through a porous rock. Our initial discussion assumes that both fluids are viscoelastic and that their behavior is described by the Noll simple material. Two Newtonian fluids are taken up as a special case.

Equations (26), (28), and (44) suggest that  $K_i$  may be a function of the characteristic viscosity and characteristic time of each phase, the surface tension, the contact angle, the characteristic length of the porous structure  $L$ , the saturation of phase  $i$ , and some measure of the local

flow in phase  $i$  such as  $|\nabla[\overline{\mathcal{P} - p_o}]^i|$ :

$$K_i = K_i(L, \mu_i, t_i, \mu_j, t_j, |\nabla[\overline{\mathcal{P} - p_o}]^i|, \sigma, \theta, s_i) \quad (46)$$

The Buckingham-Pi theorem (14) applied to Equation (46) yields

$$\frac{\mu_i}{L^2 K_i} = \beta_i = \beta_i \left( \frac{\mu_j}{\mu_i}, \frac{t_j}{t_i}, \frac{|\nabla[\overline{\mathcal{P} - p_o}]^i| L^2}{\sigma}, \frac{\sigma t_i}{L \mu_i}, \theta, s_i \right) \quad (47)$$

As discussed previously (21, 2), the most worthwhile correlation of data for the flow of viscoelastic fluids would allow the scale-up for one fluid to be accomplished on the basis of data for another fluid in a geometrically similar flow. To accomplish this we must have a specific func-

tional  $\overline{\mathbf{G}}_{\sigma^*=0}^\infty$ , which is applicable to a class of fluids; specific fluids correspond to particular values of the various parameters involved in the functional.

Since currently we rarely have a specific functional applicable to a group of fluids (with the important exception of the Newtonian fluid), we can ask how we should attempt to correlate data for two particular fluids in geometrically similar porous media. Because we are restricting ourselves to two particular fluids, the characteristic times, the characteristic viscosities, the interfacial tension, and the contact angle may all be taken to be constants. This implies that Equation (47) reduces to

$$L^2 K_i = G(|\nabla[\overline{\mathcal{P} - p_o}]^i|, L^2, L, s_i) \quad (48)$$

or

$$K_i = \overline{G}(|\nabla[\overline{\mathcal{P} - p_o}]^i|, L, s_i) \quad (49)$$

Equation (49) suggests that there is no a priori correlation which would allow one to scale up data on the basis of what happened in one particular porous medium. As one might have expected, the situation is more difficult for multiphase viscoelastic flows than for the flow of a single viscoelastic fluid (2).

As an alternative to Equation (46), one might choose to take the local volume-averaged velocity as a measure of the flow conditions in phase  $i$ :

$$K_i = K_i(L, \mu_i, t_i, \mu_j, t_j, \frac{\overline{\mathbf{v}}^i}{v}, \sigma, \theta, s_i) \quad (50)$$

By the Buckingham-Pi theorem (14) we obtain

$$\frac{\mu_i}{L^2 K_i} = \beta_i = \beta_i \left( \frac{\mu_j}{\mu_i}, \frac{t_j}{t_i}, \frac{\overline{\mathbf{v}}^i \mu_i}{\sigma}, \frac{\sigma t_i}{L \mu_i}, \theta, s_i \right) \quad (51)$$

Observing that we have restricted ourselves to two particular fluids, we may write Equation (51) as

$$L^2 K_i = G(\overline{\mathbf{v}}^i, L, s_i) \quad (52)$$

or

$$K_i = \overline{G}(\overline{\mathbf{v}}^i, L, s_i) \quad (53)$$

Like Equation (49), Equation (53) indicates that there is no a priori correlation available to allow scale-up between geometrically similar porous media.

Equations (49) and (53) suggest that, if one wishes to arrive at an obvious basis for scaling experimental data, then one must talk in terms of a specific constitutive equation for stress such as the Newtonian fluid, Equation (37). Since the characteristic time for the Newtonian fluid is zero (5), Equation (47) reduces to

$$\frac{\mu_i}{L^2 K_i} = \beta_i \left( \frac{\mu_j}{\mu_i}, \frac{|\nabla[\overline{\mathcal{P} - p_o}]^i| L^2}{\sigma}, \theta, s_i \right) \quad (54)$$

For two particular Newtonian fluids we have

$$L^2 K_i = G(|\nabla[\overline{\mathcal{P} - p_o}]^i|, L^2, s_i) \quad (55)$$

The parallels of Equations (51) and (52) are

$$\frac{\mu_i}{L^2 K_i} = \beta_i \left( \frac{\mu_j}{\mu_i}, \frac{\overline{\mathbf{v}}^i \mu_i}{\sigma}, \theta, s_i \right) \quad (56)$$

and

$$L^2 K_i = G(\overline{\mathbf{v}}^i, s_i) \quad (57)$$

To the extent that either Equation (47) or Equation (51) [for Newtonian fluids, Equations (54) and (56)] is applicable, the resistance coefficient  $K_i$  is independent of the motion of phase  $j$  and is independent of the derivatives of saturation with respect to time and position. This means that  $K_i$  may be studied under circumstances such that the saturation is maintained uniform throughout the sample. This is the standard experiment to determine the effective permeability to one fluid of a test core which is partially saturated with respect to another fluid (18, p. 57).

The quantity  $\beta_i$  in Equations (47), (51), (54), and (56) may be referred to as a relative permeability. It can be identified with  $\alpha_i$  in Equation (1) when the character-

istic length  $L$  of the porous media is taken to be the square root of the permeability of the porous media to a single Newtonian fluid.

In the usual extension of Darcy's law to multiphase flow of two specific Newtonian fluids in porous media (1, p. 653), the relative permeability for phase  $i$ ,  $\beta_i$ , is assumed to be a function of the local saturation of phase  $i$  only. In the development presented here, Equations (54) and (56) suggest that this might be true only in a limiting case such as

$$\frac{\sigma}{|\nabla[\bar{\phi} - p_o]| L^2} \rightarrow 0 \quad \text{or} \quad \frac{\sigma}{\bar{v} \mu_i} \rightarrow 0$$

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## NOTATION

- $\mathbf{f}$  = body force vector per unit mass  
 $\mathbf{F}$  = deformation gradient as defined by Equation (16)  
 $\mathbf{g}$  = acceleration of gravity  
 $\bar{H}$  = mean curvature of surface, (11, p. 205)  
 $k$  = permeability of porous media to a single-phase, Newtonian fluid  
 $K_i$  = resistance coefficient used to represent a proper value of  $K_i$  when all three proper values are identical  
 $\mathbf{K}_i$  = resistance transformation (or tensor) for phase  $i$  as defined by Equation (26)  
 $L$  = characteristic length of porous media  
 $\mathbf{n}$  = unit normal vector outwardly directed with respect to the closed surface  $S_i$   
 $\mathbf{n}_{(i)}$  ( $\mathbf{n}_{(j)}$ ) = unit normal to the phase interface directed into phase  $i$  (phase  $j$ )  
 $p$  = pressure  
 $P^i$  = average pressure associated with phase  $i$  in Equations (1) and (2); not defined precisely  
 $P_c$  = capillary pressure in Equation (2); not defined precisely  
 $\bar{\phi}$  = modified pressure as defined by Equation (23)  
 $p_{(i)}$  ( $p_{(j)}$ ) = pressure associated with phase  $i$  (phase  $j$ ) evaluated at the phase interface  
 $p_c$  = capillary pressure in Equation (44)  
 $s_i$  ( $s_j$ ) = saturation of phase  $i$  (phase  $j$ ) as defined by Equation (30)  
 $S$  = closed surface which is associated with a particular point in space  
 $S_i$  = closed bounding surface of the region  $V_i$   
 $S_{et}$  = portion of  $S_i$  formed by the intersection of phase  $i$  with  $S$   
 $S_{fi}$  = portion of  $S_i$  which is fixed in space  
 $S_{mi}$  = portion of  $S_i$  which forms a common boundary for two moving phases  
 $t$  = time  
 $t_i$  ( $t_o$ ) = characteristic time (5) of phase  $i$  when it is a Noll simple material  
 $t_p$  = characteristic time of the process  
 $\mathbf{t}$  = stress tensor  
 $\mathbf{t}_{(i)}$  ( $\mathbf{t}_{(j)}$ ) = stress tensor associated with phase  $i$  (phase  $j$ ) evaluated at the phase interface  
 $\mathbf{v}$  = velocity vector  
 $\mathbf{v}_i$  = characteristic speed of phase  $i$   
 $\frac{\bar{v}}{\bar{v}}$  = magnitude of the volume-averaged velocity associated with phase  $i$   
 $V$  = volume enclosed within closed surface  $S$   
 $V_{(n)}$  = speed of displacement (3, p. 499) of  $S_{mi}$

- $V^i$  = average velocity associated with phase  $i$  in Equation (1)  
 $V_i$  ( $V_j$ ) = volume or region of phase  $i$  (phase  $j$ ) enclosed within closed surface  $S$   
 $\mathbf{x}$  = position of a material particle at any time  $t$   
 $\mathbf{X}$  = position of a material particle in a reference configuration

## Greek Letters

- $\alpha_i$  ( $\beta_i$ ) = relative permeability of phase  $i$  in Equation (1) [Equations (47), (51), (54), and (56)]  
 $\theta$  = contact angle  
 $\mu_i$  ( $\mu_o$ ) = viscosity of phase  $i$  when fluid is Newtonian; characteristic viscosity (5) of phase  $i$  when it is a Noll simple material  
 $\rho$  = density  
 $\rho_i$  ( $\rho_j$ ) = density of phase  $i$  (phase  $j$ )  
 $\sigma$  = surface tension  
 $\tau$  = extra stress tensor (or transformation) as defined by Equation (24)  
 $\tau_{(i)}$  ( $\tau_{(j)}$ ) = extra stress tensor associated with phase  $i$  (phase  $j$ ) evaluated at the phase interface  
 $\phi$  = gravitational potential as defined by Equation (21)

## Special Symbols

- $\bar{\cdot}$  = average over  $V$  of quantity associated with phase  $i$ :  $\bar{\cdot} \equiv \frac{1}{V} \int_{V_i} \cdot dV$   
 $\langle \cdot \rangle^i$  = average over  $V_i$  of quantity associated with phase  $i$ :  $\langle \cdot \rangle^i \equiv \frac{1}{V_i} \int_{V_i} \cdot dV$   
 $*$  = dimensionless variable  
 $\text{tr } \tau$  = trace of the transformation  $\tau$

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